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SIMILARITY AND SELF-PRESERVATION IN ISOTROPIC TURBULENCE

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Measurements of the double and triple velocity correlation functions and of the energy spectrum function have been made in the uniform mean flow behind turbulence-producing grids of several shapes at mesh Reynolds numbers between 2000 and 100000. These results have been used to assess the validity of the various theories which postulate greater or less degrees of similarity or self-preservation between decaying fields of isotropic turbulence. It is shown that the conditions for the existence of the local similarity considered by Kolmogoroff and others are only fulfilled for extremely small eddies at ordinary Reynolds numbers, and that the inertial subrange in which the spectrum function varies as $k^{-\frac{5}{3}}$ (k is the wave-number) is non-existent under laboratory conditions. Within the range of local similarity, the spectrum function is best represented by an empirical function such as $k^{-a \log k}$, and it is concluded that all suggested forms for the inertial transfer term in the spectrum equation are in error. Similarity of the large scale structure of flows of differing Reynolds numbers at corresponding times of decay has been confirmed, and approximate measurements of the Loitsianski invariant in the initial period have been made. Its value, expressed non-dimensionally, decreases slowly with grid Reynolds number within the range of observation.

Turbulence-producing grids of widely different shapes are found to produce flows identical in energy decay and in structure of the smaller eddies. The largest eddies depend markedly on the grid shape and are, in general, significantly anisotropic.

Within the initial period of decay, the greater part of the energy spectrum function is self-preserving, and this part has a shape independent of the shape of the turbulence-producing grid. The part that is not self-preserving contains at least one-third of the total energy, and it is concluded that theories postulating quasi-equilibrium during decay must be considered with great caution.

Introduction

In the theory of isotropic turbulence, it has been customary to assume partial or complete similarity of the turbulent motion at all stages of the decay, that is, that the changes in the structure of the turbulence with time can be fully described by changes in magnitude of two parameters, usually a characteristic length and a characteristic velocity. This assumption is based on the notion that the rapid interchange of energy between eddies of different sizes soon leads to an equilibrium distribution of energy among the various eddy sizes, which is independent of the detailed initial conditions of the turbulence and is determined by such bulk parameters as the total energy and scale of the turbulent motion. This equilibrium may be an absolute or a quasi-equilibrium, depending on whether the turbulent transfer of energy into and out of eddies of a particular size does or does not greatly exceed the decay in intensity of these eddies. Both types of equilibrium are important in the decay of isotropic turbulence. Using these ideas, a number of workers have given accounts of the decay of isotropic turbulence and of the development and shape of the double velocity correlation function, but there has been some divergence in the type and range of the similarity assumed, and recently experimental measurements of the correlation function have led to some doubt as to the strict accuracy of the existing notions of the similarity in isotropic turbulence. A considerable amount of information about the double and triple correlations and the energy spectrum function has been collected in this laboratory, and in this paper an attempt will be made to present the experimental evidence for the various theories of similarity and to decide how far the theoretical accounts of turbulent decay agree with the observed changes in the structure of the turbulence as measured in wind-tunnels.

Some of the experimental results presented below have been published already by one or other of the authors, but the greater part of the measurements are new. Since the primary object of this paper is to assess the experimental backing of current theories of similarity and self-preservation, no attempt is made to indicate the original source of particular measurements.

Notation

ν	kinematic viscosity,
M	centre to centre spacing of rods in a square-mesh grid,
d	diameter of the cylinders composing the grid,
x	distance from the grid to the point of observation,
U	mean stream velocity in the working section of the wind-tunnel,
u_i	instantaneous value of the turbulent velocity component in the i direction,
$\overline{u^2} = \overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2}$	in isotropic turbulence. Bars indicate spatial averages,
$R_i^j(\xi_1, \xi_2, \xi_3)$	$= \overline{u_i u_j'}$, the mean value of the product of u_i at the point P, and u_j' at the point P', where PP' is a vector $\mathbf{r}$ with components ξ_1, ξ_2, ξ_3. The subscript 1 refers to the downstream direction. If there is a preferred cross-stream direction (for example, in the flow behind a parallel rod grid), the subscript 2 refers to the direction parallel to the grid elements,
$T_{ij}^k(\xi_1, \xi_2, \xi_3)$	$= \overline{u_i u_j u_k'}$, the triple velocity product under the same conditions,
$\overline{u^2} f(r)$	$= R_1^1(r, 0, 0)$ in isotropic turbulence,
λ^2	$= \overline{u^2} \Big/ \overline{\left(\dfrac{\partial u_1}{\partial \xi_1}\right)^2}$,
ϵ	total energy dissipation per unit mass,
L	$= \displaystyle\int_0^\infty f(r)\, dr$,
Λ	$= \overline{u^2} \displaystyle\int_0^\infty r^4 f(r)\, dr$,
k	radian wave-number in the spectrum of turbulence,
$E(k, t)$	energy density in the three-dimensional spectrum at wave-number k,
$\phi(k, t)$	energy density in the one-dimensional spectrum at wave-number k,
$T(k, t)$	contribution of the inertial forces to $-\dfrac{\partial E(k, t)}{\partial t}$,
$S(k, t)$	$= \displaystyle\int_0^k T(k, t)\, dk$,
k_s	$= (\epsilon/\nu^3)^{\frac{1}{4}}$.

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Local similarity

The theory of local isotropy or local similarity was originated by Kolmogoroff (1941 *a*, *b*; also Batchelor 1947), and applies only to the smaller eddies which contain little energy, and which are in an absolute equilibrium, i.e. any group of eddy sizes is receiving and losing energy at rates which are large compared with the time rate of change of the energy of eddies in the group. The existence of such a range of eddy sizes depends on observations that the eddies responsible for the final viscous dissipation of turbulent energy contain a negligible proportion of the total energy, if the Reynolds number is moderately high. Applying the generally accepted view that turbulent transfer occurs by successive instabilities of the eddies, any eddy being unstable with respect to formation of eddies an order of magnitude smaller and so on, it seems likely that sufficiently far into the range of eddies in absolute equilibrium, the energy transfer is due only to transfer of energy from eddies also in absolute equilibrium. Under these conditions, neither the amount nor the distribution of the energy among the large eddies can have any influence on the distribution of energy among the various small eddy sizes, and the distribution can only depend on the rate of energy transport, on the size of the eddies considered, and on the viscosity. The distribution function can be expressed in as many ways as there are ways of specifying eddy size, but the most generally accepted function is the three-dimensional spectrum function $E(k)$, defined by

$$E(k) = \frac{k^2}{2\pi^2}\iiint R_i^i(\mathbf{r})\,\frac{\sin kr}{kr}\,d\tau(\mathbf{r}),$$

where $d\tau(\mathbf{r})$ is the element of volume about $\mathbf{r}$ (Batchelor 1949). While the notion that a single eddy size is represented by a single component of the spectrum differs slightly from the intuitive physical notion of an eddy, this representation seems to be a reasonable compromise between mathematical and experimental convenience and the physical concept of a single eddy. Then, applying dimensional analysis to the absolute equilibrium range, it follows that

$$E(k) = \epsilon'^{\frac{1}{4}}\nu^{\frac{5}{4}}\chi(\epsilon'^{-\frac{1}{4}}\nu^{\frac{3}{4}}k), \tag{1}$$

where ϵ' is the energy transfer into the low k limit of the absolute equilibrium range, and χ is a universal function, not dependent on the mode of production of the turbulence. Under the special condition that viscosity is negligible, a condition only possible at very high Reynolds numbers for not too large values of k,

$$E(k) = A\epsilon'^{\frac{2}{3}}k^{-\frac{5}{3}}, \tag{2}$$

where A is an absolute constant. In this particular case, ϵ' is clearly the total energy dissipation ϵ, since all the energy dissipation is due to eddies whose wave-numbers are greater than those of the eddies satisfying this relation. When no such range exists, it is questionable what value of ϵ' should be used in equation (1).

To find the form of χ, it is necessary to make some assumption about the nature of the transfer of turbulent energy from one wave-number to another. If the ordinary Kármán-Howarth (1938) correlation equation is transformed to the three-dimensional spectrum form, it becomes

$$\frac{\partial E(k,t)}{\partial t} + T(k,t) = -2\nu k^2 E(k,t), \tag{3}$$

where $T(k,t)$ is a transform of the triple velocity correlation, and surmises about its form are necessary before the equation can be solved. Since $T(k,t)$ represents the net effect of energy transfer on $\dfrac{\partial E(k,t)}{\partial t}$, it is better to consider $S(k,t)$, the rate of transfer of energy from all wave-numbers less than k to all wave-numbers greater than k. Then, neglecting $\dfrac{\partial E(k,t)}{\partial t}$,

$$\frac{\partial S(k,t)}{\partial k}+2\nu k^2E(k,t)=0. \tag{4}$$

Three specific hypotheses have been made about $S(k,t)$.

(*a*) Heisenberg (1948*a*, *b*) regards the action of the small eddies as equivalent to an eddy viscosity, and puts

$$S(k,t)=2\eta(k)\int_0^k k'^2E(k',t)\,dk',$$

where

$$\eta(k)=K\int_k^\infty \sqrt{\left(\frac{E(k')}{k'^3}\right)}dk'$$

and derives the spectrum

$$E(k)=\left(\frac{8\epsilon}{9K}\right)^{\frac{2}{3}}k^{-\frac{5}{3}}\left[1+\left(\frac{k}{k_1}\right)^4\right]^{-\frac{4}{3}}, \tag{5}$$

where $k_1=\left(\dfrac{3K^2\epsilon}{8\nu^3}\right)^{\frac{1}{4}}$. For large k, $(k \gg k_1)$, this becomes

$$E(k)=K^2\frac{\epsilon^2}{16\nu^4}k^{-7}.$$

It is interesting to note that any assumption of an eddy viscosity due only to eddies of wave-numbers greater than k leads to a similar result. For example, a very general form for $\eta(k)$ may be used,

$$\eta(k)=\sum_c K_c\left[\int_k^\infty\left(\frac{E}{k'}\right)^{1/2c}\frac{dk'}{k'}\right]^c,$$

where c takes only positive values if the integrals are to converge, and the K_c's are pure numbers. When k is so large that

$$\int_k^\infty k'^2E\,dk' \ll \int_0^\infty k'^2E\,dk'$$

equation (4) becomes

$$\frac{\epsilon}{\nu}\frac{\partial\eta(k)}{\partial k}+2\nu k^2E=0. \tag{6}$$

Substituting the general form for $\eta(k)$ in this equation, an integral equation for $E(k)$ is obtained, whose solution is easily verified to be

$$E(k)=\left(\sum_c c^2K_c\right)^2\frac{\epsilon^2}{16\nu^4}k^{-7},$$

i.e. the modified expression for $\eta(k)$ still leads to a power-law spectrum with the same exponent.

(*b*) Obukhov (1941), by analogy with the expression for turbulent energy production in shear flow, puts

$$S(k,t)=\alpha\left[\int_0^k k'^2E(k')\,dk'\right]^{\frac{1}{2}}\int_k^\infty E(k')\,dk',$$

but this leads to a solution increasing with k for large values of k, which is physically impossible.

(*c*) Kovasznay (1948) has assumed that $S(k,t)$ depends only on $E(k,t)$ and k:

$$S(k,t) = \beta[E(k)]^{\frac{3}{2}}\, k^{\frac{5}{2}}.$$

The solution is

$$E(k) = \beta^{-\frac{7}{3}}\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}\left[1-\left(\frac{k}{k_2}\right)^{\frac{4}{3}}\right]^2,$$

where

$$k_2 = 2^{\frac{3}{4}}\beta^{-\frac{1}{2}}\left(\frac{\epsilon}{\nu^3}\right)^{\frac{1}{4}}.$$

This spectrum has a high-frequency cut-off.

In general, any attack by supposing that $S(k,t)$ may be written as the product of two terms, one involving integrals over all eddies with wave-number less than k, and the other all eddies with wave-number greater than k, leads to a power-law spectrum at very large k, irrespective of the exact form of the factors. This arises from the essential constancy of the term representing the effect of the larger eddies when k is sufficiently large. The objection to a power-law spectrum is that sufficiently high-order moments of the spectrum become infinite, with consequent infinities in the corresponding derivatives of the correlation function, and this is contrary to the intuitive physical notion that the phenomena are completely smooth, and that the associated functions are analytic.

Self-preservation of the energy-containing eddies

The local similarity described by Kolmogoroff is a similarity equally valid for all types of fully-developed turbulence, but, by its nature, it cannot apply to the energy-containing eddies whose structure depends on more parameters than the energy transport and the kinematic viscosity. It is usual to assume that the structure of these eddies remains similar during decay, but that the structure, expressed in terms of a scale length and a scale velocity, is not necessarily the same for other decaying turbulent fields. To avoid confusion, this class of hypotheses will be called self-preservation hypotheses, and the term similarity used to describe that class of hypotheses in which a scale length and a scale velocity alone are sufficient to determine the structure of any field of turbulence.

The most comprehensive self-preservation hypothesis is to assume that the whole of the structure of the turbulence remains similar during decay, but Batchelor (1948) has shown theoretically that complete self-preservation cannot occur except as an asymptotic condition at very large decay times, when the spectrum function becomes

$$E(k,t) = Ck^4\, \mathrm{e}^{-2\nu t k^2}.$$

This asymptotic form has been verified experimentally for flow far behind a square-mesh grid at fairly low Reynolds numbers (Batchelor & Townsend 1948), and represents a state in which there is no transfer of energy between eddies of different sizes, and in which each component of the spectrum decays independently. The motion has little resemblance to ordinary turbulent flow, and the solution, while interesting in itself, has no direct bearing on the problem of the structure of isotropic turbulence at finite Reynolds numbers.

While complete self-preservation is only possible under these special non-typical conditions, limited self-preservation is theoretically possible over a wide range of conditions, and current theories of energy decay postulate self-preservation over finite ranges of

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wave-number, the exact range depending on the view taken of the nature of the physical processes involved in turbulent transfer. Both for very small wave-numbers and for very large wave-numbers, there is known to be similarity, for when k is very small

$$E(k,t) = Ck^4,$$

where C is invariant during decay (Batchelor 1949), and when k is very large it is almost certain that local similarity is true. Existing theories assume self-preservation during decay of the energy-containing part of the spectrum plus either very large wave-numbers or very small wave-numbers. Explicitly, the hypotheses are

(*a*) that $E(k,t) = E_0\psi(k/k_s)$ for $k > k_c$, where k_c is such that $\int_0^{k_e} E(k,t)\,dk \ll \int_0^\infty E(k,t)\,dk$ and E_0, k_s are functions of time, and

(*b*) that $E(k,t) = Ck^4\mathscr{F}(k/k_s)$ for all $k < k_r$, where k_r is such that $\int_{k_r}^\infty E(k,t)\,dk \ll \int_0^\infty E(k,t)\,dk$.

The first hypothesis (Heisenberg 1948*b*) supposes that transfer of energy from one wave-number to another leads to a quasi-equilibrium which at sufficiently high wave-numbers becomes an absolute equilibrium of the type described in the theory of local similarity. The form of the quasi-equilibrium spectrum will depend on the Reynolds number of the turbulence, but it can be shown that the hypothesis leads to the 'linear' decay law

$$\overline{u^2} \propto t^{-1}$$

valid in the initial period of decay, by considering the spectrum equation. Integrating this equation,

$$S(k',t) + \frac{\partial}{\partial t}\int_0^{k'} E(k,t)\,dk = -2\nu\int_0^{k'} k^2 E(k,t)\,dk.$$

The requirements for self-preservation of $S(k,t)$ and $E(k,t)$ are that

$$E(k,t) = E_0(t)\,\psi(k/k_s),$$

$$S(k,t) = [E_0(t)]^{\frac{3}{2}}\,k_0^{\frac{5}{2}}\eta(k/k_s),$$

and, putting $k' = \alpha k_s$

$$\alpha^{\frac{5}{2}}E_0^{\frac{3}{2}}\,k_s^{\frac{5}{2}}\eta(\alpha) + \frac{d}{dt}(E_0 k_s)\int_0^\alpha \psi(x)\,dx = -2\nu k_s^3 E_0\int_0^\alpha x^2\psi(x)\,dx.$$

Since $\eta(x)$ and $\psi(x)$ are invariant during decay, it follows

(i) that $$E_0 \propto k_s, \text{ and}$$

(ii) that $$\frac{d}{dt}(k_s^2) \propto k_s^4,$$

i.e. $$k_s^2 \propto t^{-1}.$$

Then the intensity of the turbulence is

$$\overline{u^2} = \int_0^\infty E(k,t)\,dk = E_0 k_s\int_0^\infty \psi(x)\,dx$$
$$\propto t^{-1}.$$

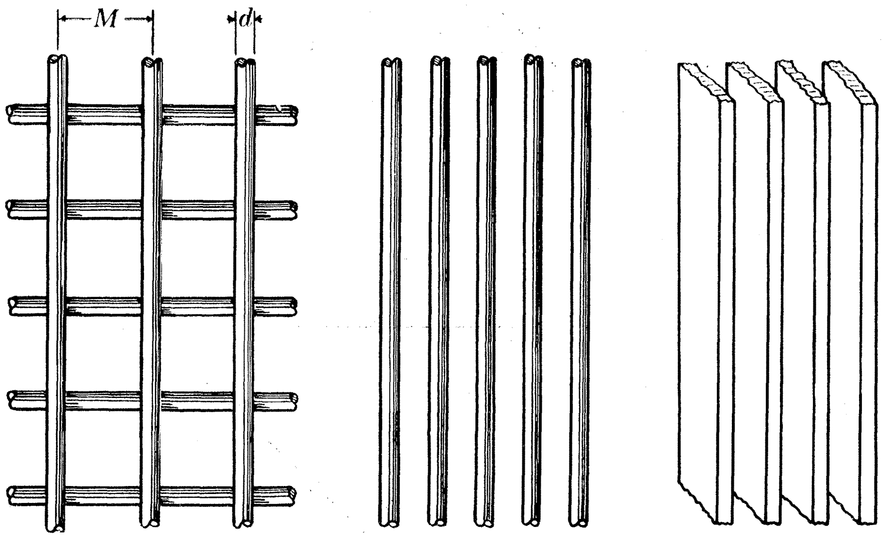
The second hypothesis (see Kármán & Lin 1949) is based on the apparent similarity in shape of all correlation curves at different Reynolds numbers and times of decay if the initial parabolic region is neglected. Physical reasons for this type of similarity are not obvious, for the rate of transfer of energy from eddies of wave-number less than k to those of wave-number greater than k becomes negligible as k approaches zero, even considered as a fraction of the total energy in these wave-numbers, and, in fact, the invariant Ck^4 range is a reflexion of this disappearance of the inertial transfer term. In the absence of inertial transfer of energy between the range and the energy-containing range, it is difficult to see how the invariant range can impose a scale-velocity relation on the main body of the spectrum. However, the hypothesis has been used by Kolmogoroff (1941 *b*) and Frenkiel (1948) to deduce the decay law

$$\overline{u^2} \propto t^{-\frac{10}{7}}.$$

This decay law is not valid in the initial period, but Kármán & Lin (1949) have suggested that it may be true in the intermediate period of decay (Batchelor & Townsend 1948 *a*).

Experimental arrangements

Most of the measurements were made in the small wind-tunnel in the Cavendish Laboratory, but a few measurements of double and triple correlations at high Reynolds numbers were made in the low-turbulence wind-tunnel at the National Physical Laboratory,



square mesh grid   parallel cylinder grid   slats grid

Figure 1. Turbulence-producing grids used in the experiments.

Teddington. The measurements in the Cavendish Laboratory were made in the turbulent flow behind bi-plane grids of circular cylinders of mesh-diameter ratio $\frac{16}{3} = 5{\cdot}33$, except for a few measurements using grids of parallel cylinders, and a grid of parallel prisms of elongated rectangular section (figure 1). The measurements in the National Physical Laboratory were behind a bi-plane grid of cylinders $\frac{15}{16}$ in. diameter, spaced centre to centre $6\frac{15}{16}$ in., i.e. a mesh-diameter ratio of $\frac{37}{5} = 7{\cdot}4$.

A hot-wire anemometer was used for the detection of the turbulent velocity fluctuations, and the associated electrical measuring equipment is very similar to that already described (Batchelor & Townsend 1948 *a*). The only additions are:

(*a*) an improved form of the circuit for the measurement of triple correlations, which, while preserving the economy in non-linear circuit elements, contrives to balance the statistical fluctuations of the output so that more accurate readings are possible (Stewart 1951), and

(*b*) a zero-beat heterodyne frequency analyzer for measurement of spectra.

The experimental results fall into several groups:

(*a*) longitudinal double velocity correlation functions,

(*b*) longitudinal triple velocity correlation functions,

(*c*) spectra of the longitudinal velocity fluctuations,

(*d*) decay measurements behind grids of various shapes.

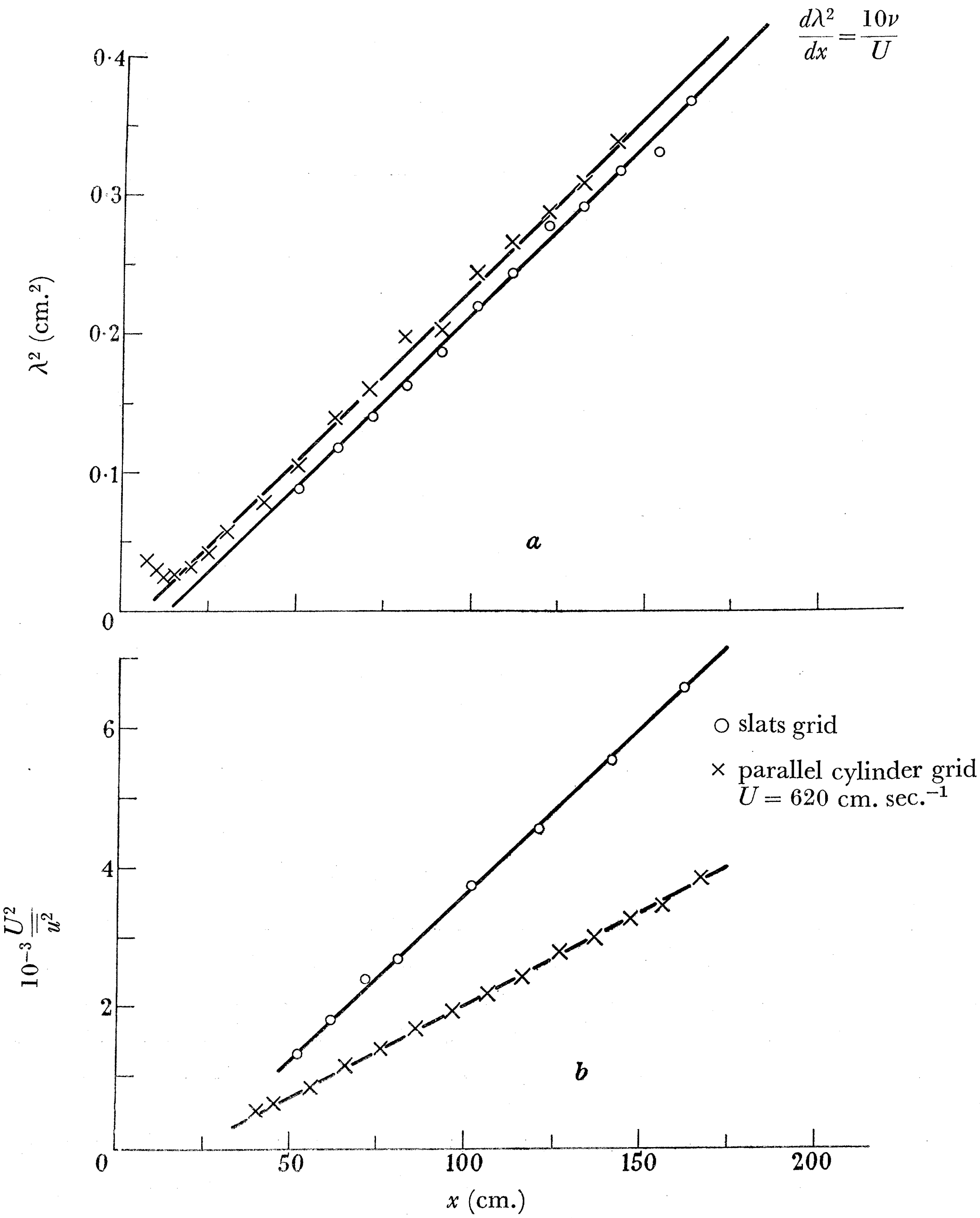


FIGURE 2. Decay of turbulence behind grids of differing shapes.

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The influence of grid geometry on the turbulence

All theories of isotropic turbulence assume that the turbulence behind a uniform grid can be fully described in terms of comparatively few parameters, and that the exact geometry of the grid elements is unimportant. For square-mesh grids of circular cylinders, the decay can be described using as parameters the mean stream velocity and the effective scale $C_d M$

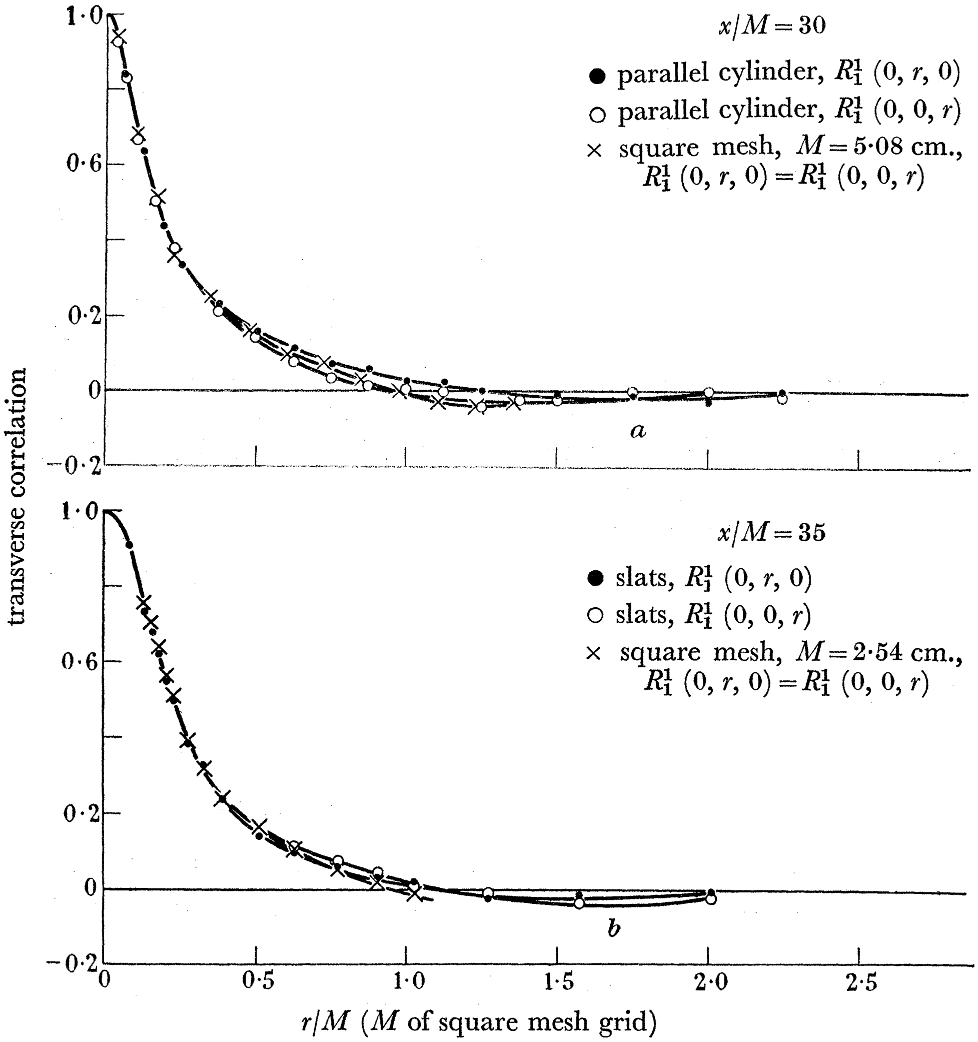

Figure 3. Transverse correlation functions behind grids of differing shapes.

(C_d is the drag-coefficient of the grid) and it has been assumed that the turbulence behind any grid can be described in terms of two similar parameters. To determine the extent to which this assumption is valid, measurements have been made of the turbulence behind two grids, both considerably different from the usual bi-plane square grid. One was made of parallel circular cylinders, of diameter 0·95 cm. and uniformly spaced 2·54 cm. centre to centre, while the other consisted of parallel rectangular prisms, of section 3·7 by 0·4 cm., spaced 1·9 cm. apart (figure 1), looking rather like an open Venetian blind. At various distances downstream, measurements were made of the intensities of the turbulent velocity components, of λ^2, and of the transverse double velocity correlations in directions both parallel with and perpendicular to the grid elements, i.e. $R_1^1(0, r, 0)/\overline{u_1^2}$ and $R_1^1(0, 0, r)/\overline{u_1^2}$.

Except very close to the grids, the turbulence was found to be isotropic in the sense that

$$\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2}$$

and the energy decay law was the linear law, found to be valid in the initial period for bi-plane grids by Batchelor & Townsend (1948*a*), i.e.

$$\overline{u^2} \propto (t-t_0)^{-1}, \quad \lambda^2 = 10\nu(t-t_0)$$

as shown in figure 2. In figure 3*a* and *b*, the correlation curves for the two grids are compared with selected correlation curves measured behind a bi-plane grid. The turbulence behind the parallel cylinder grid of 2·54 cm. periodic spacing is found to resemble very closely the turbulence behind a bi-plane grid of 5·08 cm. spacing, and correlation functions for these grids are compared in figure 3*a*, where M is the periodic spacing of the bi-plane grid, i.e. 5·08 cm. Similarly in figure 3*b*, the slats grid is compared with a bi-plane grid of $M = 2{\cdot}54$ cm., and again M is the periodic spacing of the bi-plane grid, in this case 2·54 cm. For reasonably small values of r, the correlation curves are independent of the nature of the grid, but at larger values of r, the departures from true isotropy become evident, and these departures persist to the limit of the available decay times. Moreover, by transforming the correlations to the three-dimensional spectrum function, it can be shown that the range of wave numbers strongly affected by the grid geometry contains about one-fifth of the total energy. Since the energy decay law and the spectrum at large wave-numbers are nearly independent of grid shape, it must be concluded that neither are sensitive to the shape of the spectrum at small wave numbers. Provided that these small wave-numbers are excluded, it is clear that grid shape is irrelevant to the structure of the turbulence behind a grid, and that ordinary bi-plane square-mesh grids produce turbulence essentially similar to that found behind these rather unusual grids.

Energy spectrum of isotropic turbulence

Since it is not possible to measure directly the three-dimensional spectrum function $E(k)$, the measurements are of the longitudinal spectrum function $\phi(k)$, which is related to $E(k)$ by

$$E(k) = k^2 \frac{\partial^2 \phi}{\partial k^2} - k \frac{\partial \phi}{\partial k}$$

and represents the one-dimensional spectrum of a velocity fluctuation component taken along a line parallel to the direction of the component. It is, except for a constant scale factor, identical with the frequency spectrum of the fluctuations recorded by a hot-wire (Taylor 1938). That is, if $\psi(p)\,dp$ is the energy contained between radian frequencies p and $p+dp$

$$\phi(k) = U\psi(p),$$

where

$$Uk = p$$

and U is the mean velocity.

The spectrum analyzer used in these experiments could be adjusted to have band-widths of either 20 or 120 hertz. The narrow band-width was used for measurements below 300 hertz. For measurements at the high-frequency end of the spectrum, it is undesirable to put the whole output of the amplifier into the analyzer as it would overload before appreciable output were obtained, so, taking advantage of the availability of differentiating circuits, the high-frequency spectrum was studied by measuring the spectrum, not of u, but of $\partial u/\partial t$, $\partial^2 u/\partial t^2$, and $\partial^3 u/\partial t^3$. These spectra are related by

$$\psi(u,p) = p^{-2}\psi\left(\frac{\partial u}{\partial t}, p\right) = p^{-4}\psi\left(\frac{\partial^2 u}{\partial t^2}, p\right) = p^{-6}\psi\left(\frac{\partial^3 u}{\partial t^3}, p\right),$$

where $\psi(e,p)$ is the spectrum of the quantity $e(t)$. Unfortunately, it is difficult to determine accurately the band-width of the analyzer, and for this reason, the absolute values of the spectrum functions are rather uncertain, but the relative values are believed to be fairly accurate. The electrical characteristics of the equipment are adequate over the range of use, but the response of the hot-wire may be falling at high frequencies due to the finite length of the wire. While it is difficult to be certain, this effect is believed to be less than the scatter of the individual observations, partly because hot-wires of different lengths gave substantially identical results, and partly because of the consistency of the results at different times of decay and Reynolds numbers.

The greater part of the results have been obtained at a wind-speed of 620 cm. sec.$^{-1}$, using square-mesh grids of mesh 0·635, 1·27, 2·54 and 5·08 cm., and the results for the various spectra are plotted in figures 4 to 7, in a non-dimensional plot, using characteristic wave-number

$$k_s = (\epsilon/\nu^3)^{\frac{1}{4}}$$

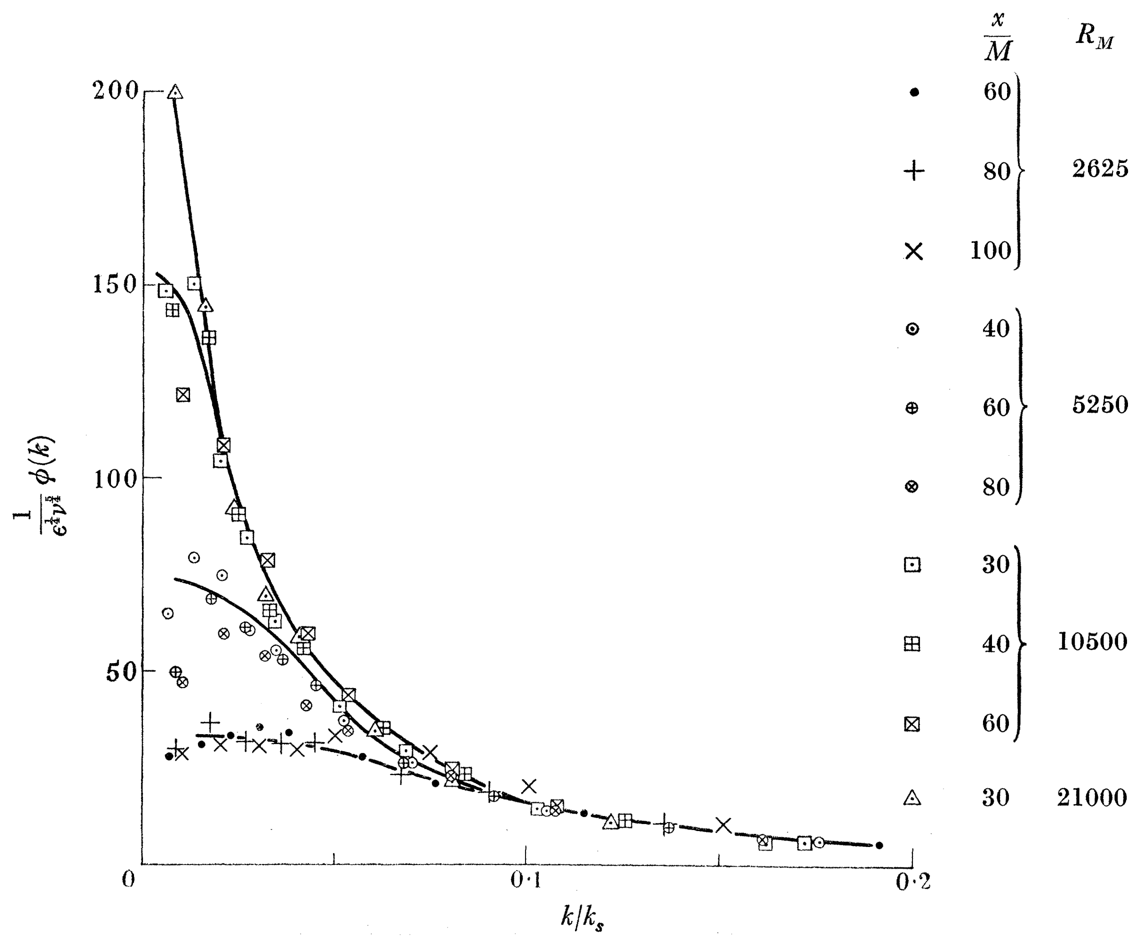

FIGURE 4. Spectrum functions of u fluctuations.

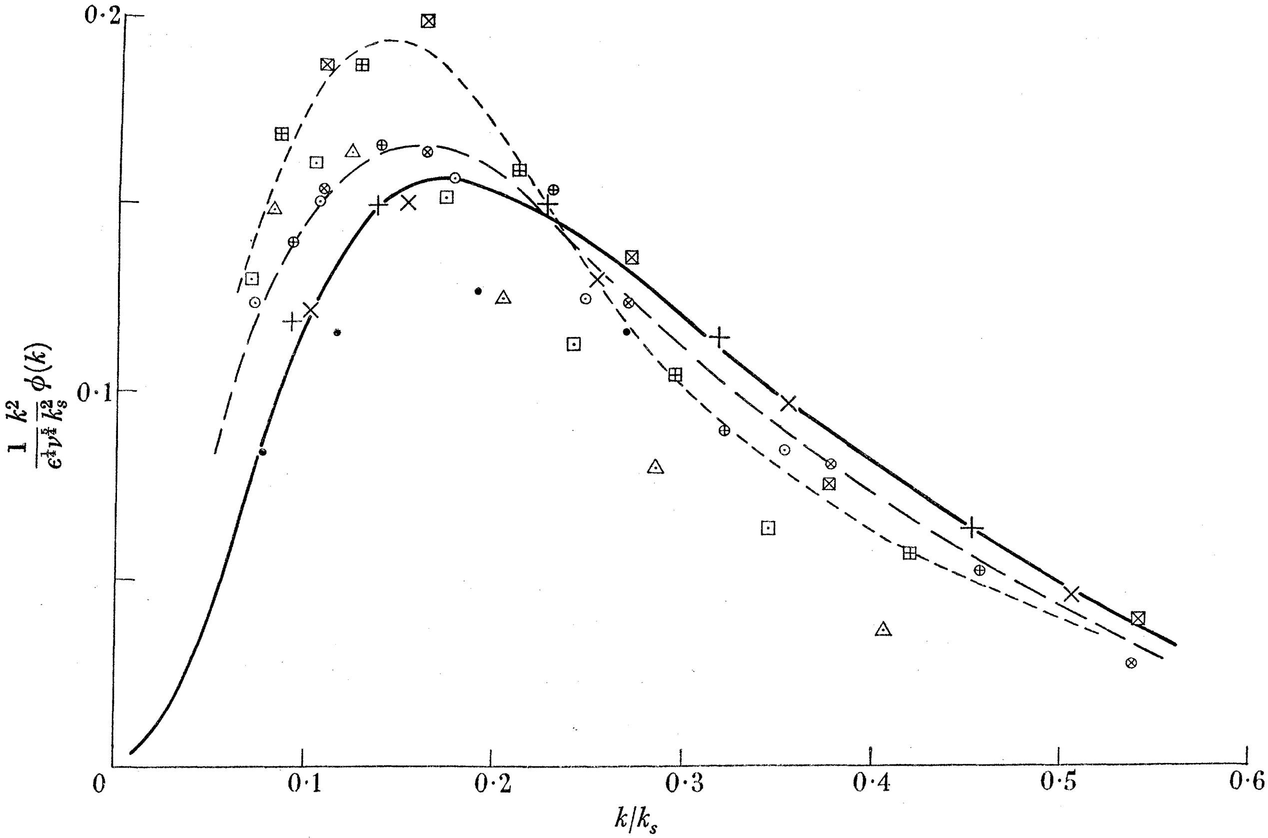

FIGURE 5. Spectrum functions of $\partial u/\partial t$ fluctuations.
—$R_M = 2625$; ---$R_M = 5250$; - - -$R_M = 10500$.

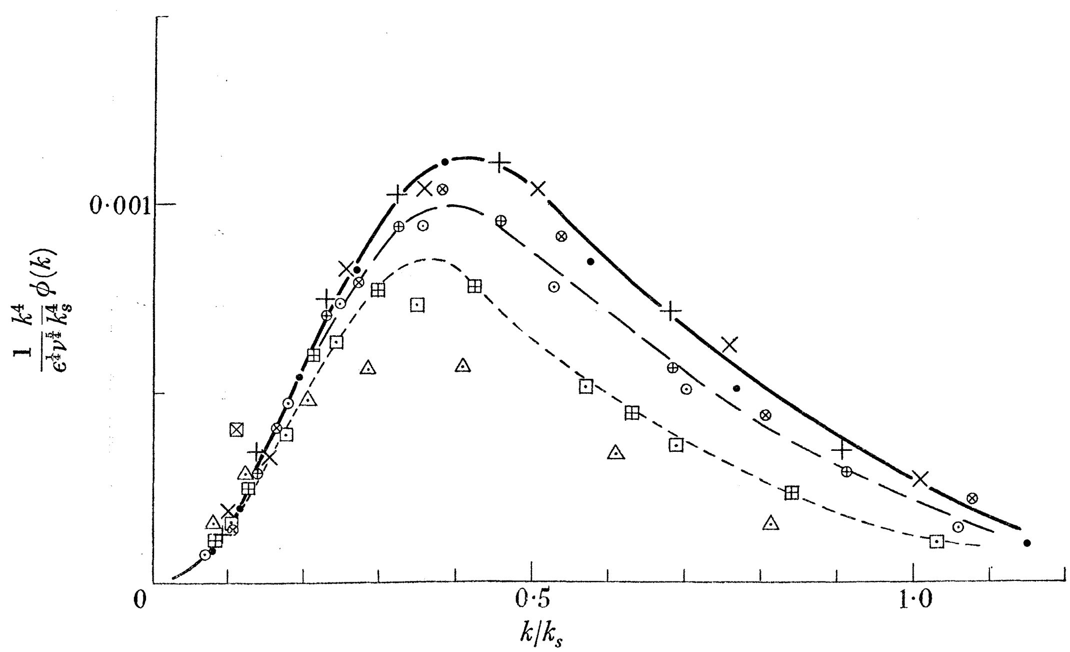

FIGURE 6. Spectrum functions of $\partial^2 u/\partial t^2$ fluctuations.
—$R_M = 2625$; ---$R_M = 5250$; - - -$R_M = 10500$.

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and characteristic spectral intensity $\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}$.* For the purposes of this plot, ϵ' was assumed to be equal to the total dissipation ϵ. Using the mean decay law (Batchelor & Townsend 1948*a*)

$$\frac{U^2}{\overline{u^2}} = D\frac{x-x_0}{M},$$

$$\epsilon = \frac{1\cdot 5}{D}\left(\frac{M}{x-x_0}\right)^2\frac{U^3}{M} = 15\nu\overline{\left(\frac{\partial u_1}{\partial \xi_1}\right)^2},$$

where D and x_0/M are nearly absolute constants. Experimentally, $D = 135$ and $x_0/M = 10$.

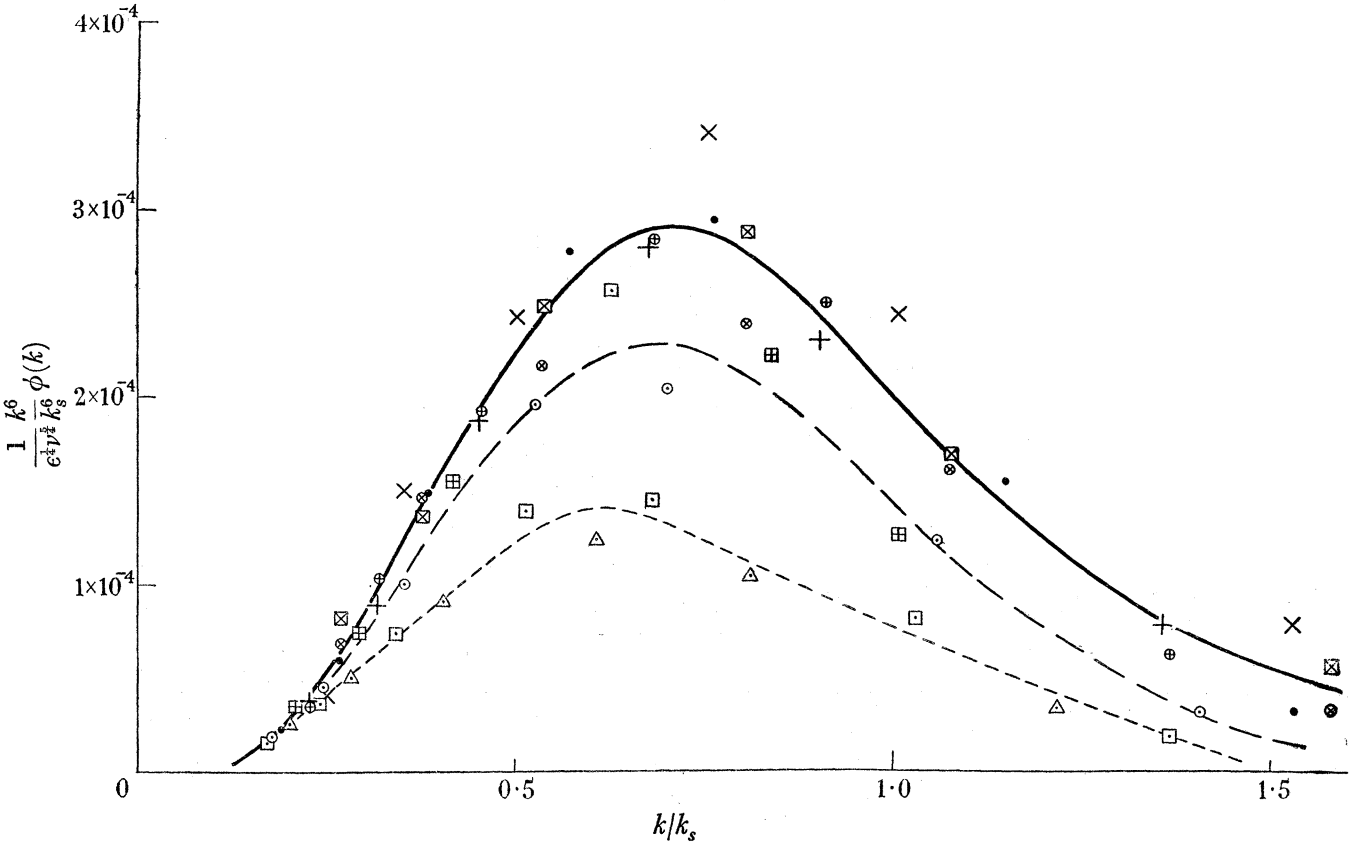

FIGURE 7. Spectrum functions of $\partial^3 u/\partial t^3$ fluctuations.
—x/M = 60, 80, 100; ---x/M = 40; ---x/M = 30.

It is clear from the results that there is no part of the spectrum for which both the contribution to the total energy and the contribution to $\overline{(\partial u_1/\partial \xi_1)^2}$ are negligible, so theoretical justification for the use of the total energy dissipation is lacking and the success of the assumption is its main justification. This identity of ϵ and ϵ' is consistent with the notion that the large eddies enter into the transfer term in the form $\int_0^k k'^2 E(k')\,dk'$. It should also be noted that, if the spectrum is self-preserving during decay, then, for any one mesh Reynolds number, this method of plotting will lead to a single spectrum irrespective of local similarity (for example, see figure 4). Considering the high-frequency end of the spectrum first, it is apparent that for k/k_s greater than about 0·6, and at not too small values of x/M, the spectra are all substantially identical in this form. Below $k/k_s = 0\cdot 6$, there are small but significant systematic

* Numerical tables of the observed spectrum functions are given in the appendix.

differences between the spectra at different mesh Reynolds numbers, these differences being most obvious in the spectra of $\partial u_1/\partial \xi_1$, but sufficiently clear in the spectra of $\partial^2 u_1/\partial \xi_1^2$. It must be concluded that, at the Reynolds number range of these experiments, the conditions appropriate to the setting-up of the equilibrium spectrum only occur for k greater than $0{\cdot}6k_s$. (Since the three-dimensional spectrum function $E(k)$ depends on the behaviour of $\phi(k)$ only in the immediate neighbourhood of k, these conclusions apply equally to $E(k)$. The differences are the more significant since the method of plotting ensures that, independently of any similarity, the area under the spectrum of $\partial u_1/\partial \xi_1$, i.e. $\overline{(\partial u_1/\partial \xi_1)^2}$, is independent of the mesh Reynolds number.

At low values of x/M, the intensities present in the high-frequency part of the spectrum are considerably less than normal, and this effect cannot be dismissed as experimental error, since observations at larger values of x/M but similar values of the total energy dissipation are not affected in this way. The obvious explanation is that initially the turbulence is produced in the form of large eddies, and the small eddies represented by the high wave-number components of the spectrum are formed by a complicated sequence of interactions between eddies. If the time necessary to form eddies corresponding to wave-numbers near $0{\cdot}5k_s$ is of order $20M/U$, then until $x/M \gg 20$, the initial conditions will still influence the form of the large wave-number part of the spectrum. The negative slope of the λ^2 against x/M curve (figure 2) at very small x/M is consistent with this point of view.

The original spectrum measurements of Simmons & Salter (1938) and Dryden (1938) seemed to show that, at sufficiently high Reynolds numbers, the spectrum shape was nearly independent of mesh Reynolds number $R_M = MU/\nu$, and of decay time, and characterized by the turbulent intensity $\overline{u^2}$ and the longitudinal scale L. The present measurements show that, at any one mesh Reynolds number, the self-preservation of spectrum shape is good for all but the smallest values of the wave-number, and that the appropriate scale is $(\nu^3/\epsilon)^{\frac{1}{4}}$ or λ rather than L. When L is used the agreement is naturally better near $k = 0$, but worse over most of the spectrum. The apparent similarity found by Simmons & Salter, and by Dryden is due mostly to the lack of freedom in the possible monotonic functions satisfying the conditions

$$\phi(0) = \frac{2}{\pi} L\overline{u^2},$$

$$\int_0^\infty \phi(k)\, dk = \overline{u^2}$$

and has no other significance. As will be seen later, study of the correlation function shows that the spectrum at small values of k is very far from self-preserving during decay.

Self-preservation as a function of r/λ

The spectra in figures 4 to 7 shows to what extent the small eddies are self-preserving in form during decay. Larger scales of motion, however, are best studied by means of the correlation function, as measurements of correlation are more accurate than measurements of spectra in this region and departures from self-preservation are more easily seen. Thus, the failure of self-preservation for large eddies is very evident in figures 8 and 9, in which the double and triple correlations are plotted against r/λ. The triple correlation appears rather more sensitive than the double to this variation, but the measurement of triple correlations at large values of r is difficult, and it would be unwise to rely too much on the exact form of the curves beyond the extrema.

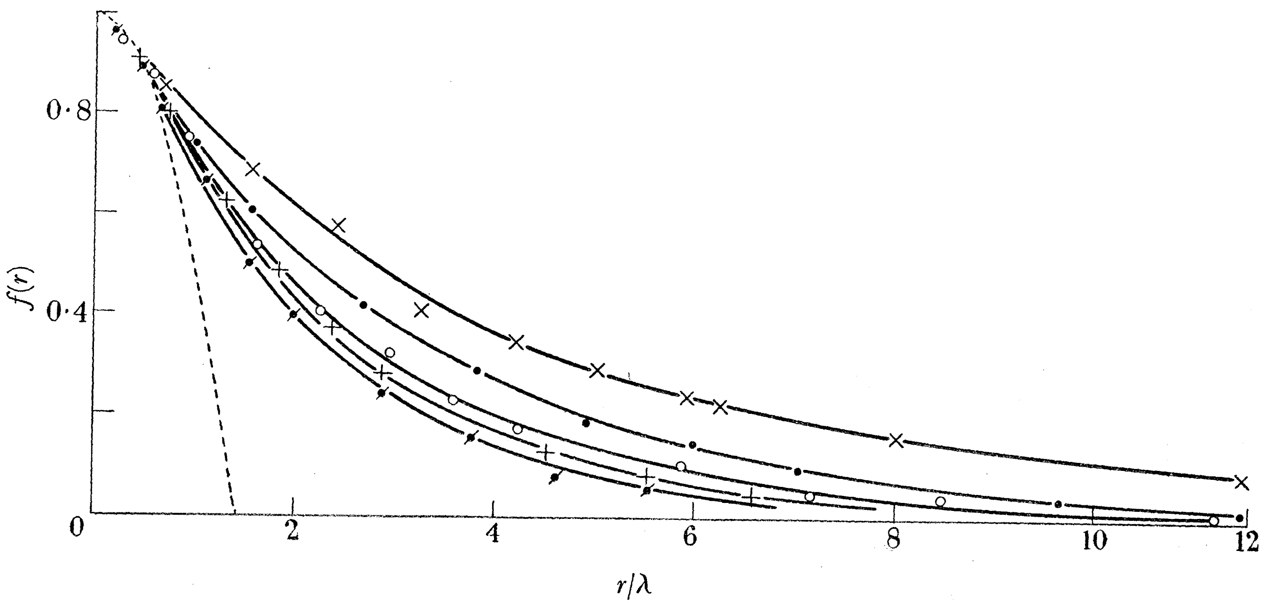

FIGURE 8. Double velocity correlation function during decay of turbulence at $R_M = 5300$. x/M: ×, 20; ●, 30; ○, 60; +, 90; ⌀, 120; ---, $1 - \frac{r^2}{2\lambda^2}$.

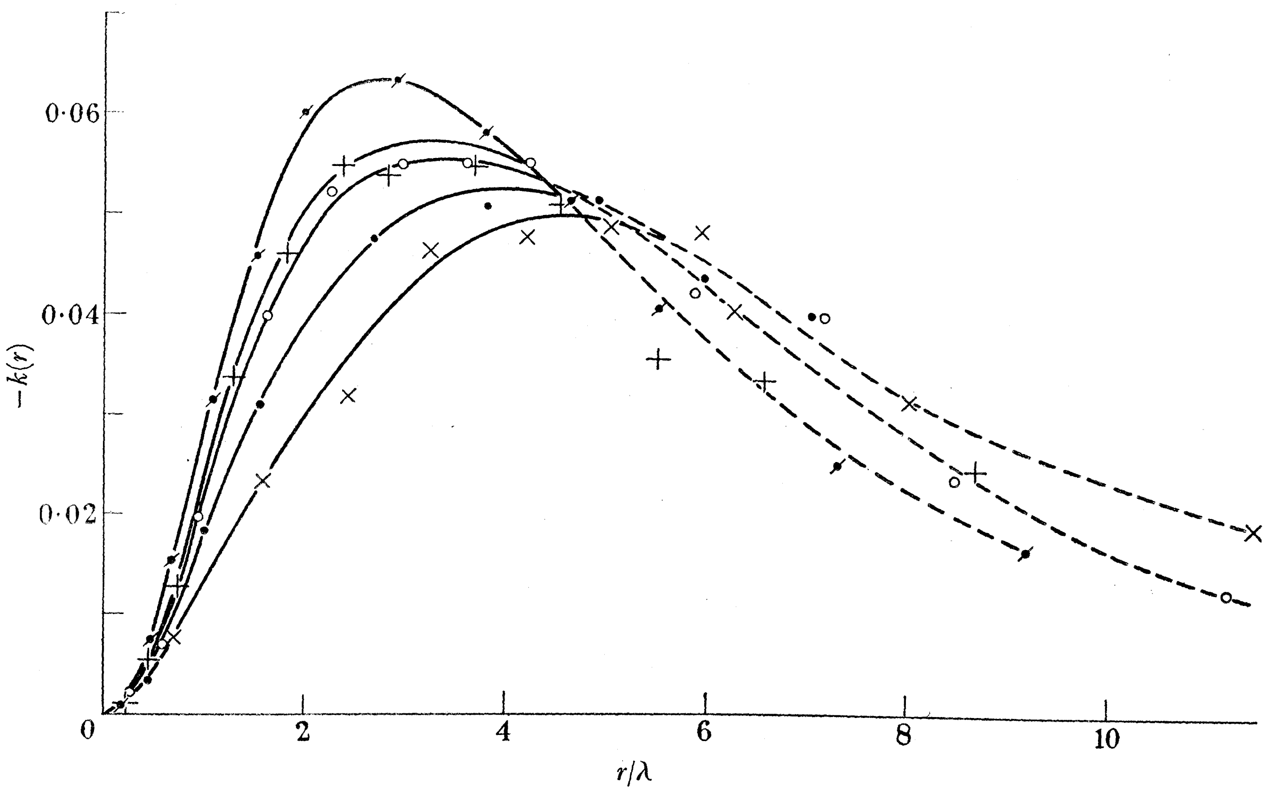

FIGURE 9. Triple correlations during decay of turbulence at $R_M = 5300$. x/M: ×, 20; ●, 30; ○, 60; +, 90; ⌀, 120.

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If the double correlations of figure 8 are transformed to the three-dimensional spectrum function, the differences in the spectra at different decay times, plotted against $k\lambda$ or k/k_s, extend to wave-numbers considerably beyond the maximum in the spectrum. The small wave-number parts of these spectra, plotted in such a way that they coincide at large values of k, are shown in figure 10. It thus appears that a significant proportion of the total energy may be outside the self-preserving region of the spectrum even while the initial period decay law is closely obeyed.

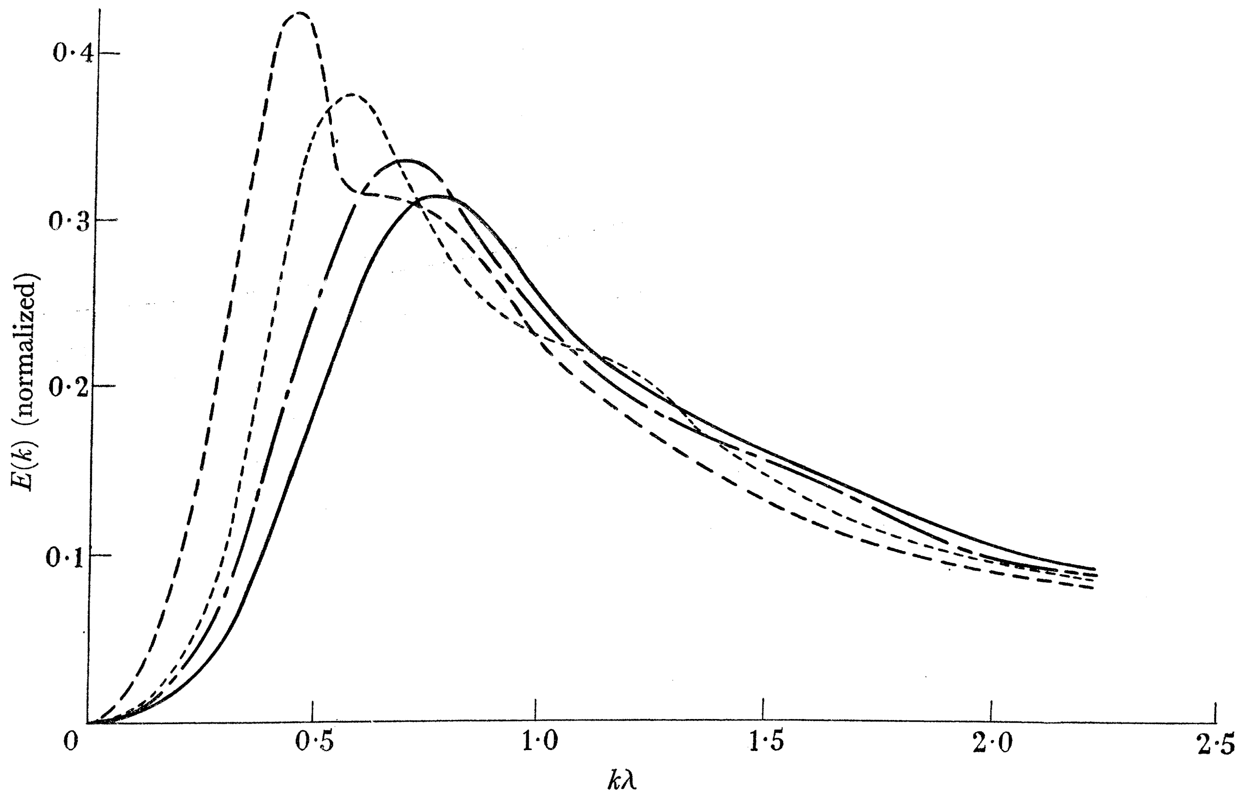

FIGURE 10. Three-dimensional spectra at $R_M = 5300$. x/M: —, 120; ———, 90; ---, 60; – – –, 30.

STRUCTURE OF THE LARGE EDDIES DURING DECAY

The correlation equivalent of the invariance during decay of the coefficient of k^4 in the three-dimensional spectrum function is the well-known Loitsianski (1939) invariant relation:

$$\overline{u^2}\int_0^\infty r^4 f(r) = \Lambda,$$

where Λ is independent of time. In the initial period of decay,

$$\overline{u^2} \propto (t-t_0)^{-1}$$

and so

$$\int_0^\infty r^4 f(r)\, dr \propto (t-t_0).$$

If there is self-preservation of the correlation function at large values of r, such that $f(r,t) = f(r/r_0(t))$, then $r_0(t) \propto (t-t_0)^{\frac{1}{5}}$.

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Figure 11 shows the same correlation curves as figure 8, but this time they are plotted against $\frac{r}{M}\left(\frac{x-x_0}{M}\right)^{-\frac{1}{5}}$, where x_0 is the virtual origin of the turbulence determined from the energy decay curve. The parabolas $(1-\frac{1}{2}r^2/\lambda^2)$ have been drawn for each curve to show how widely they must separate at small values of r. At large values of r, there appears to be no systematic variation among the curves, and this is not accidental for with a comparatively small change in either the value of x_0 or of the power $-\frac{1}{5}$ the agreement disappears. It seems, then, that there is a range of large eddies that are self-preserving in a quite different way

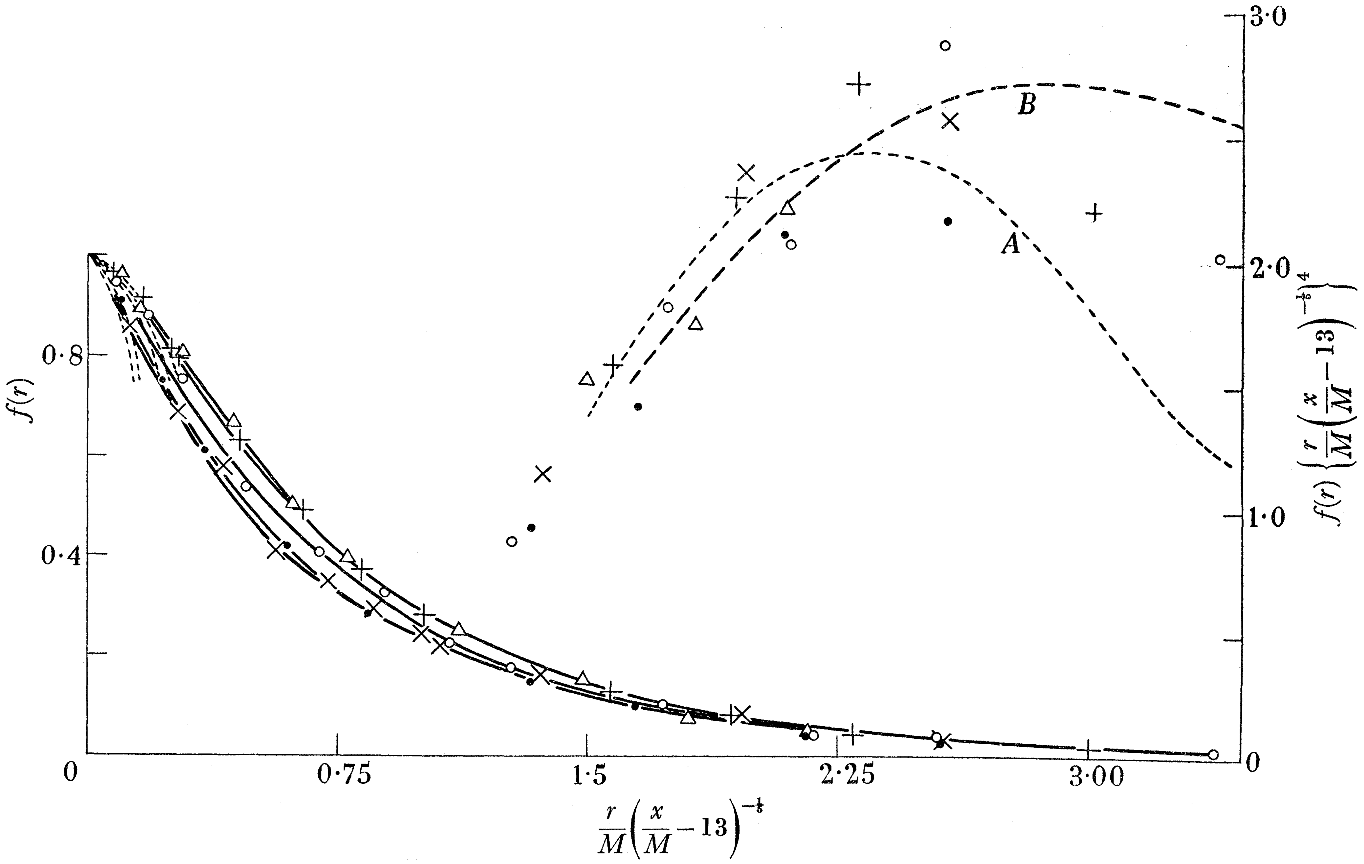

FIGURE 11. Self-preservation of the correlation function at large values of r. $R_M = 5300$. x/M: ×, 20; ●, 30; ○, 60; +, 90; △, 120. A, $Ar^4\,e^{-\alpha t^2}$; B, $Br^4\,e^{-br}$.

from the small eddies, i.e. they have a characteristic length that varies as $t^{\frac{1}{5}}$ rather than $t^{\frac{1}{2}}$. The three-dimensional spectrum assigns about 15% of the total energy to this range, so it is not negligible. The limited duration of the initial period of turbulent energy decay is almost certainly due to the existence of this group of large eddies of appreciable energy whose dimensions increase comparatively slowly. Initial period decay comes to an end when the smaller energy-containing eddies have increased so much in size that they are no longer distinct from these more slowly growing large eddies. The scale of the turbulence then increases more slowly, and correspondingly the rate of energy decay decreases less rapidly than was the case in the initial period.

The curve $f(r)\left(\frac{r}{M}\right)^4\left(\frac{x-x_0}{M}\right)^{-\frac{4}{5}}$ has also been drawn on figure 11. If the initial period decay law is strictly true, the area under this curve should be the same at all decay times, and the observed variation is small for the portion of the curve accessible to observation. At the largest values of r for which any sort of accuracy is possible, the correlation is decreasing (as r increases) slightly faster than an exponential but not so fast as a Gaussian error function, and it is possible to estimate the value of Λ. This may be a dubious procedure, but, considering the present lack of information on the magnitude of Λ and the obvious interest of this invariant, the attempt is worth while. In figure 12, values of Λ obtained by assuming (i) that $f(r) \propto e^{-ar}$ and (ii) that $f(r) \propto e^{-br^2}$, for r greater than measurable values, are plotted against mesh Reynolds number in the non-dimensional form Λ/M^5U^2. The actual values of Λ almost certainly lie between these two extremes. Figure 12 also includes values of Λ/M^5U^2 based on the data of Batchelor & Townsend (1948*b*) on the decay in the final period for $R_M = 650$, 950 and 1360. The values for $R_M = 950$ and 1360 have been recalculated using a slightly different, and probably more accurate, method than that used in the original paper, i.e. tangents of slope $4\nu M/U$ have been drawn to the λ^2 against x/M curves to obtain the final period virtual origin, rather than drawing tangents of arbitrary slope to the energy curves.

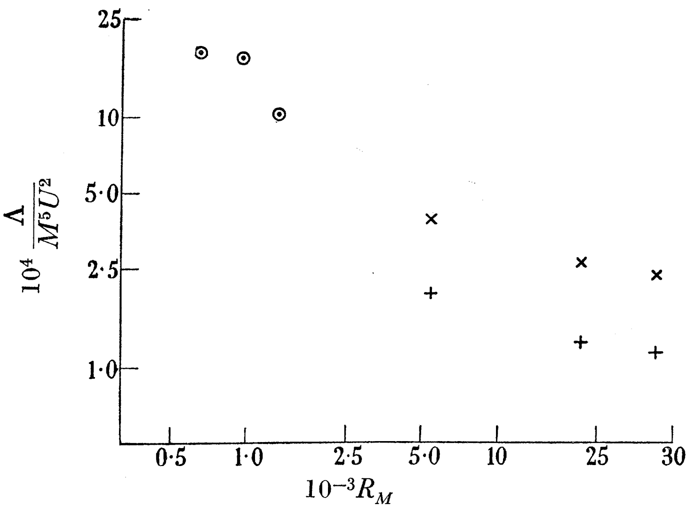

FIGURE 12. Variation of the Loitsianski invariant Λ with R_M; $\odot$, calculated from decay in the final period; $\times$, calculated assuming $f(r) \propto e^{-at}$ at large r; $+$, calculated assuming $f(r) \propto e^{-bt^2}$ at large r.

If the correlation curve has an ultimate high Reynolds number shape, and if $\overline{u^2}$ varies as U^2 at constant x/M, Λ/M^5U^2 will be independent of Reynolds number at sufficiently high Reynolds number. At the Reynolds numbers used in this laboratory, Λ/M^5U^2 appears to decrease with increasing Reynolds number. This point will be considered below.

Similarity of large eddy structure at different Reynolds numbers

It has long been known that the velocity correlation function, $f(r)$, measured at a fixed distance down-stream of a grid at various tunnel speeds is nearly independent of tunnel speed except at very small values of r, i.e. provided that λ is small compared with lengths characteristic of the energy-containing eddies, for example, with the integral scale L. At values of R_M for which λ/L is not small, the correlation is everywhere greater than at the higher Reynolds numbers, and this departure from similarity of the correlation function may be attributed to the significant proportion of the energy-containing eddies which are directly influenced by viscosity when the Reynolds number is low. At sufficiently high Reynolds numbers, the eddies directly influenced by viscosity contain negligible energy, and the correlation function should be independent of Reynolds number and cusp at $r = 0$. While it is not practicable to attain sufficiently high Reynolds numbers to confirm this prediction, it is possible to recover the asymptotic correlation at large values of r from observed correlations by the following method. Because the effect of viscosity is to absorb energy directly from the smaller eddies and so to reduce the energy transfer to yet smaller eddies, at low Reynolds numbers the proportion of the total energy in eddies of less than a given size is less than at high Reynolds numbers. As the velocity product $R_1'(r, 0, 0)$ is determined mainly by eddies larger in size than r, it is clear that the ratio of this velocity product to the total intensity, that is, the correlation function $f(r)$, is greater than the asymptotic value at high Reynolds numbers in the ratio $(1+\Delta)$, where Δ is a measure of the relative lack of energy in the small eddies due to the direct action of viscosity. To a first approximation, $\Delta = \alpha\lambda^2/M^2$, where α is a function of x/M only, and the test of these ideas is whether $f(r)/(1+\Delta)$ at sufficiently large values of r/M is independent of mesh Reynolds number.

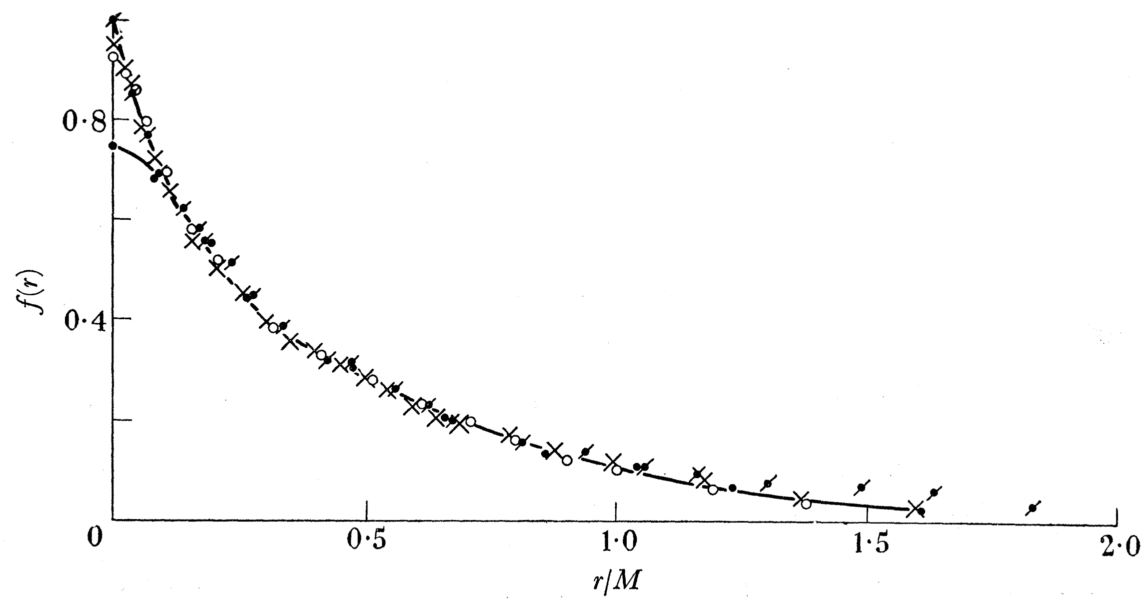

FIGURE 13. Adjusted double correlations.
$x/M = 30$. R_M: ●, 5250; ○, 21000; × 42000; ø, 115000.

The results of this process are shown in figure 13, where $f(r)/(1+\Delta)$ is plotted against r/M for a series of mesh Reynolds numbers. Δ is obtained using

$$\Delta = \alpha\lambda^2/M^2 = 10\alpha(x/M)\, R_M^{-1}.$$

There is only one adjustable constant, α, which has been set to be 6·1. An indication of the improvement in fit is given by the magnitude of the correction factor applied to the correlation curve for $R_M = 5250$. If this correction were not applied, the correlations would be consistently 30% greater than those at larger Reynolds numbers, and the points would lie well away from the high Reynolds number results. There are two sets of results, the first series being obtained for square-mesh bi-plane grids at $R_M = 5300$, 21200 and 42400, all at $x/M = 30$, and the second series a single correlation curve at $R_M = 115000$. In the first series, the M/d ratio was 5·33, in the second 7·4. The second correlation function is therefore not strictly comparable with the first series of measurements, and differences are to be expected at very large values of r/M where the correlation depends more critically on grid geometry. It also means that an effective mesh length must be chosen, and this has been done to obtain best agreement near $f(r) = 0{\cdot}5$. If the similarity of the large eddies is real, then not only should the adjusted double correlations agree at large r/M, but similarly adjusted triple correlations should also fall along a single curve (figure 14). Due to difficulty in the absolute calibration of the circuit for the measurement of triple correlations, all the measurements in any particular determination of the triple correlation function may be in error by up to $\pm 5\%$, but there is distinct, if not conclusive, evidence of similarity of the triple correlation function at large r.

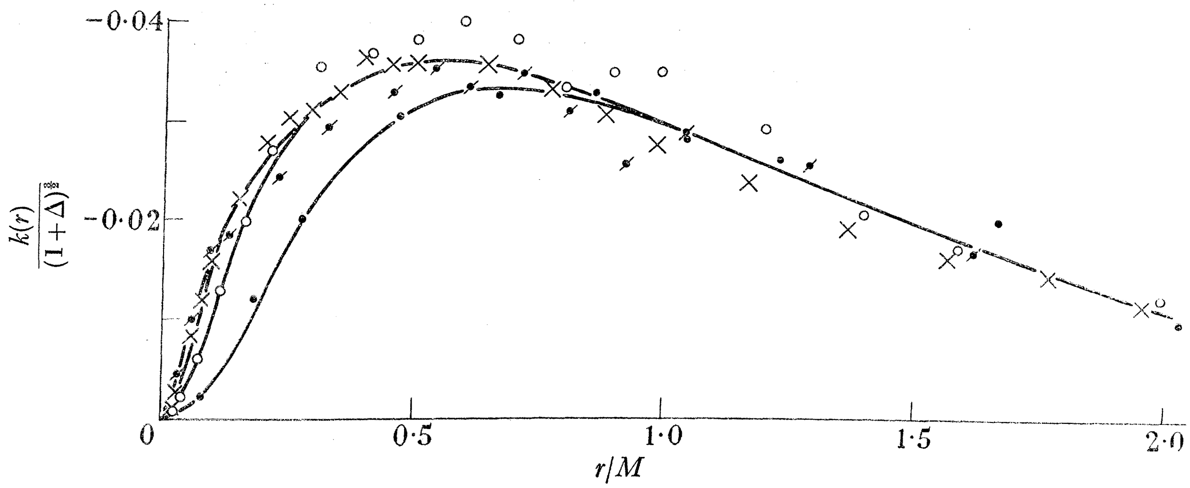

FIGURE 14. Adjusted triple correlations. $x/M = 30$. R_M: ●, 5300; ○, 21200; ×, 42400; ●̸, 115000.

These adjusted forms of the double and triple correlation functions probably form an adequate approximation to the asymptotic forms of these functions at very large Reynolds numbers. The observed systematic decrease with increasing Reynolds number of the invariant Λ/M^5U^2 is partially accounted for by the variation of $f(r)$ (unadjusted) with Reynolds number, but there is also a variation due to the fact that neither $\overline{u^2}/U^2$ nor $\overline{u^2}/U^2(1+\Delta)$ are completely independent of R_M. This difference is probably due to variation of the drag coefficient of the grid at these comparatively low Reynolds numbers.

Composite spectrum of turbulence

Direct measurement of the spectrum is the most accurate method of determining the spectral distribution of turbulent energy at large wave-numbers, but at very small wave-numbers correlation measurements are more reliable. In an intermediate range, both are sufficiently accurate, and it is possible to obtain a composite spectrum of considerable accuracy over the whole range of wave-number by transforming the correlation function and combining it with the direct measurements (figure 15). There are significant differences between the spectrum function and the transformed correlation function at the smallest wave-numbers, due in part to the necessary approximation that

$$\frac{\partial}{\partial x} = -\frac{1}{U}\frac{\partial}{\partial t}.$$

Such large-scale motions may well change appreciably in the time taken for them to pass the hot-wire. The exact interpretation of the spectrum function at these wave-numbers is also uncertain, for it is very likely that inhomogeneity of the turbulence in the downstream direction influences the structure of the largest eddies, and that they do not correspond closely with large eddies in a decaying homogeneous field of turbulence.

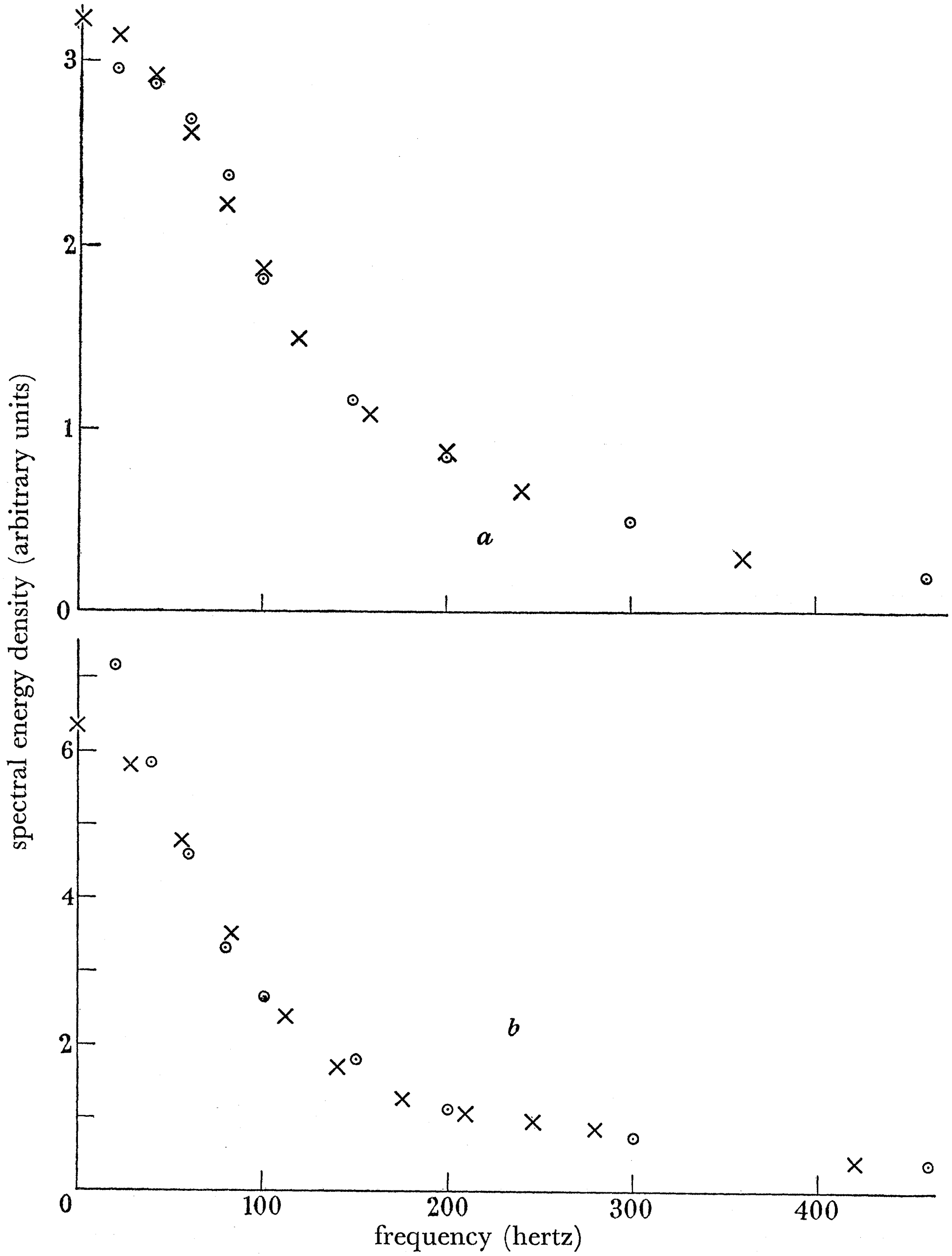

FIGURE 15. One-dimensional spectra: ⊙, direct measurement; ×, transformed double correlation.

	M (cm.)	x/M	U (cm./sec.)	R_M
a	1·27	60	620	5250
b	5·08	30	620	21000

LOCAL SIMILARITY

Kolmogoroff's hypothesis of local similarity is an analytical description of the generally accepted physical picture of the degradation of turbulent energy by a cascade series of instabilities, and its consequent plausibility and the promise it gives of permitting the application of information gained from the study of isotropic turbulence to the more complex problem of shear flow have given the theory a prominent place in recent study of turbulence. The spectrum measurements reported above make it clear that the portion of the spectrum which conforms to the requirements of the theory is limited to extremely high wave-numbers under conditions easily obtainable in wind-tunnels. It is not difficult to see that this result is consistent with the necessary condition for establishment of local similarity in the viscous range:

$$-\frac{\partial E}{\partial t} \ll 2\nu k^2 E.$$

Making use of the result that for sufficiently large k, $E(k, t)$ is self-preserving during decay,

$$E(k, t) = k_s \psi(k/k_s),$$

where
$$k_s = (\epsilon/\nu^3)^{\frac{1}{4}} = 15^{\frac{1}{4}} R_\lambda^{\frac{1}{2}}/\lambda.$$

Then
$$-\frac{\dfrac{\partial E}{\partial t}}{2\nu k^2 E} = \frac{5}{2\sqrt{15}} \frac{1+n}{\alpha^2 R_\lambda},$$

where
$$n = -\frac{k}{E}\frac{dE}{dk}$$

and $\alpha = k/k_s$. The condition for absolute equilibrium is then that

$$\frac{5}{2\sqrt{15}} \frac{1+n}{\alpha^2 R_\lambda} \ll 1$$

and at $\alpha = k/k_s = 0{\cdot}6$, $R_\lambda = \sqrt{(\overline{u^2})}\, \lambda/\nu = 30$, $n = 6$,

$$\frac{5}{2\sqrt{15}} \frac{1+n}{\alpha^2 R_\lambda} = 0{\cdot}42,$$

which is not particularly small. At $\alpha = 1$, the ratio is $0{\cdot}12$, and the similarity condition is satisfied. This is in fair agreement with the experimental result that a universal spectrum form is only found for $k > 0{\cdot}6 k_s$, and this small portion of the spectrum function represents all that is known of the equilibrium range.

It is now clear why non-dimensional parameters of the turbulence such as

$$S_0 = \overline{\left(\frac{\partial u_1}{\partial \xi_1}\right)^3} \Big/ \left[\overline{\left(\frac{\partial u_1}{\partial \xi_1}\right)^2}\right]^{\frac{3}{2}} \quad \text{and} \quad D_2 D_4^{-2} D_6 \quad \left[D_n = \int_0^\infty k^n E(k)\, dk\right]$$

vary with Reynolds number up to the limit attainable in the wind-tunnel (Batchelor & Townsend 1949; Stewart 1951), for they are determined largely by the part of the spectrum outside the range of absolute equilibrium. Only at Reynolds numbers so high that these parameters are determined solely by the equilibrium spectrum will they become absolute constants characteristic of motion in conditions of local similarity.

A number of workers (Kolmogoroff 1941 *a*; Heisenberg 1948 *a*; Batchelor 1947; Townsend 1948) have attempted to find evidence of the inertial subrange of the equilibrium spectrum by fitting $E = Ak^{-\frac{5}{3}}$ to part of the spectrum, or, what is equivalent, fitting $f(r) = 1 - A'r^{\frac{2}{3}}$ to correlation curves. The present work shows directly that no significantly extensive region of the spectrum can be described in this way, and, indirectly, that the condition that inertial forces shall dominate the spectrum is incompatible with the condition for absolute equilibrium at these Reynolds numbers. At sufficiently high Reynolds numbers, these conditions are no longer incompatible, and Proudman (1951), using the Heisenberg form for the transfer term $S(k, t)$, has shown that an appreciable inertial subrange appears if $R_\lambda > 500$. The assumption of a particular form for $S(k, t)$ should not affect the result greatly, and, indeed, if the condition for an inertial subrange is written as

$$k/k_s < 0{\cdot}1,$$

combining this with the condition for absolute equilibrium

$$\frac{k^2}{k_s^2} > 10 \times \frac{5}{2\sqrt{15}} \frac{1+n}{R_\lambda}$$

leads to $$R_\lambda > 1730 \quad (\text{since } n = \tfrac{5}{3}),$$

which is of the same order of magnitude. It might be mentioned that $R_\lambda = 500$ corresponds to a mesh Reynolds number of $R_M = 2800000$, nearly ten times the highest reported value.

While the extremely small extent of the observed equilibrium spectrum prevents the full application of the theory, a sufficient range has been observed to discriminate between the various suggested forms of the transfer term $S(k, t)$. The spectra show that there is no sharp high-frequency cut-off, and that the value of $n = -\dfrac{d(\log E)}{d(\log k)}$ increases continuously to a value in excess of ten. Thus none of the suggested forms of the transfer term are valid in the equilibrium range, and it is concluded that no power-law is likely to represent the asymptotic form of the spectrum at large wave-numbers. It is more likely that the equilibrium spectrum is approximately described by $E \propto k^{-a \log k}$, when k is sufficiently large, i.e. n increases logarithmically with k.

Quasi-equilibrium

Although the range for which there is absolute equilibrium is very small and plays little part in the decay of turbulence, there is considerable evidence that a very important part of the turbulent energy is in a state of moving or quasi-equilibrium. The theory of the quasi-equilibrium was first developed by Heisenberg (1948 *b*), using the same form for the inertial transfer term as he used in the range of absolute equilibrium. Since the predictions of this theory are not valid at very large wave-numbers, it may be that the detailed conclusions are not valid elsewhere in the spectrum, but the general concept of quasi-equilibrium is still of value.

The observation that turbulence from such different sources as a square-mesh grid and the 'slats' grid is similar in its small eddy properties suggests that there must be strong tendencies toward a common form at large wave-numbers. Similarity breaks down for $k < 0{\cdot}6k_s$, but the spectrum function is self-preserving over a much larger region, containing over half the energy and nearly all the viscous dissipation (figures 4 and 5). Thus two parameters in addition to ν are sufficient to determine a very large part of the initial period spectrum. ϵ, the dissipation, is enough to define the similarity spectrum, but the quasi-equilibrium self-preserving part of the spectrum depends also on another parameter which varies with R_M but not with x/M. A suitable parameter is $R_\lambda = \sqrt{(\overline{u^2})}\,\lambda/\nu$, which is virtually constant during the initial period (Batchelor & Townsend 1948*a*).

The fact that the initial period decay law is the same as would be expected if the whole spectrum were self-preserving leads one to conclude that the quasi-equilibrium range that *is* self-preserving plays a dominant role in the initial period behaviour. It appears that once this quasi-equilibrium state is set up the energy drawn from the small wave-numbers is kept just sufficient to maintain it, in spite of the fact that as the decay time increases the distribution of energy in the small wave-numbers changes so that relatively more and more is found at larger wave-numbers (figure 10). On the other hand the particular values of the two parameters that determine the quasi-equilibrium must be determined originally by the distribution of energy in the large eddies. Thus the particular kind of quasi-equilibrium that is set up is determined by the initial conditions in the small wave-numbers, but once set up it appears to have an internal stability that makes it very insensitive to changes in these larger scales of motion.

The initial period of decay cannot continue indefinitely, however, for with increasing decay time the non-self-preserving large eddies encroach more and more on the quasi-equilibrium range. One of two effects may bring the period to a close. Either the viscous dissipation of the large eddies will become significant relative to the dissipation in the quasi-equilibrium range, or the distribution of energy in the small wave-numbers will become so unsuitable that the energy drawn from them by the self-preserving range can no longer be maintained at the correct amount for a linear decay. Information regarding the end of the initial period as a function of R_M is not very reliable, and unfortunately the working section of the Cavendish wind-tunnel is too short to be suitable for this purpose. What facts are available have been collected by Batchelor & Townsend (1948*a*), and indicate that the decay time, in terms of x/M, at which the linear decay law breaks down does not vary greatly with R_M. This supports the view that the end of the period is caused by a crisis in the transfer of energy from small to large wave-numbers rather than by a relative increase in the direct viscous dissipation from the large eddies.

Table of conclusions

For convenience, the various theories of isotropic turbulence, together with notes on their experimental backing, are tabulated (table 1).

We wish to thank Mr A. Fage for permission to carry out experimental work in the non-turbulent wind-tunnel of the National Physical Laboratory, and to acknowledge the willing co-operation and help given by Mr L. F. G. Simmons and his staff during our stay there.

One of us (R.W.S.) was in receipt of a scholarship from the Research Council of Ontario while these experiments were being conducted. We are also indebted to the Aeronautical Research Committee for a grant towards the cost of the turbulence-measuring equipment.

TABLE 1

general theory	form of theory	assumption	prediction	experimental evidence
local similarity	—	cascade decay and equilibrium of small eddies	$E(k) = \epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}\chi(k/k_s)$ for k large $E(k) \propto \epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$ for $k \ll k_s$	$E(k) = \epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}\chi(k/k_s)$ for $k > 0{\cdot}6k_s$ no $k^{-\frac{5}{3}}$ range for $R_\lambda < 150$
	eddy viscosity transfer term	$S(k, t) = 2\eta(k)\int_0^k k'^2E(k', t)\, dk'$ $\eta(k) = K\int_k^\infty \sqrt{\frac{E(k')}{k'^3}}\, dk'$	$E(k) \propto k^{-7}$ as $k \to \infty$	$-\frac{d \log E(k)}{d \log k} > 7$ for $k > k_s$
	shear transfer term	$S(k, t) = \alpha\left[\int_0^k k'^2E(k')\, dk'\right]^{\frac{1}{2}}\int_k^\infty E(k')\, dk'$	$E(k, t) \to \infty$ as $k \to \infty$	physically impossible
	local transfer term	$S(k, t) = \beta[E(k, t)]^{\frac{3}{2}}\, k^{\frac{5}{2}}$	$E(k) = \beta^{-\frac{4}{3}}\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}[1-(k/k_2)^{\frac{4}{3}}]^2$ large wave-number cut-off	no sign of any cut-off
complete self-preservation	—	only possible if transfer is negligible	$E(k, t) = Ck^4\, e^{-2\nu tk^2}$ $\overline{u^2} \propto t^{-\frac{5}{2}}$ for R_λ small	$E(k, t) = Ck^4\, e^{-2\nu tk^2}$ $\overline{u^2} \propto t^{-\frac{5}{2}}$ for $R_\lambda < 5$
limited self-preservation	$E(k, t) = E_0\psi(k/k_s)$ for $k > k_e$	quasi-equilibrium becoming absolute for large k	$\overline{u^2} \propto t^{-1}$ if $\int_0^{k_e} E(k)\, dk \ll \int_0^\infty E(k)\, dk$	$E(k, t) = E_0\psi(k/k_s)$ for $k > 0{\cdot}1k_s$ $\overline{u^2} \propto t^{-1}$ initial period $\int_0^{0{\cdot}1\,k_s} E(k)\, dk \ll \int_0^\infty E(k)\, dk$
	$E(k, t) = Ck^4\mathscr{F}(k/k_0)$ for $k < k_r$	apparent similarity of correlations except at very small r	$\overline{u^2} \propto t^{-\frac{10}{7}}$ if $\int_{k_r}^\infty E(k)\, dk \ll \int_0^\infty E(k)\, dk$	not valid in initial period; no evidence on validity in intermediate period

Appendix. Experimental measurements of spectrum function

$R_M = 2625$, $x/M = 60$

$\frac{k}{k_s}$	$\frac{\phi(k)}{\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\frac{k^2\phi(k)}{k_s^2\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\frac{k^4\phi(k)}{k_s^4\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ ($\times 10^3$)	$\frac{k^6\phi(k)}{k_s^6\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ ($\times 10^4$)
0·0077	27	—	—	—
0·0153	31	—	—	—
0·0230	33	—	—	—
0·0306	36	—	—	—
0·0383	34	—	—	—
0·0575	28	—	—	—
0·077	21	0·084	0·75	—
0·115	13·2	0·115	1·95	—
0·192	5·3	0·126	5·44	2·27
0·268	—	0·115	8·3	5·95
0·383	—	0·068	11·1	14·8
0·575	—	—	8·5	27·6
0·766	—	—	5·1	29·3
1·15	—	—	1·0	15·4
1·53	—	—	—	3·2

$R_M = 2625$, $x/M = 80$

$\frac{k}{k_s}$	$\frac{\phi(k)}{\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\frac{k^2\phi(k)}{k_s^2\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\frac{k^4\phi(k)}{k_s^4\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ ($\times 10^3$)	$\frac{k^6\phi(k)}{k_s^6\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ ($\times 10^4$)
0·0091	30	—	—	—
0·0181	36	—	—	—
0·0272	32	—	—	—
0·0362	31	—	—	—
0·0453	32	—	—	—
0·0679	23·5	—	—	—
0·091	18·4	0·118	1·26	—
0·136	10·9	0·149	1·46	—
0·226	3·8	0·149	7·49	3·66
0·317	—	0·114	10·2	8·95
0·452	—	0·063	11·1	18·8
0·680	—	—	7·1	28·0
0·91	—	—	3·5	23·0
1·36	—	—	—	7·8

$R_M = 2625$, $x/M = 100$

$\frac{k}{k_s}$	$\frac{\phi(k)}{\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\frac{k^2\phi(k)}{k_s^2\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\frac{k^4\phi(k)}{k_s^4\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ ($\times 10^3$)	$\frac{k^6\phi(k)}{k_s^6\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ ($\times 10^4$)
0·0101	28	—	—	—
0·0202	30	—	—	—
0·0303	30	—	—	—
0·0404	29	—	—	—
0·0505	33	—	—	—
0·076	29	—	—	—
0·101	21	0·121	1·86	—
0·152	11	0·149	3·32	—
0·252	3·5	0·129	8·3	4·1
0·354	—	0·096	10·4	15·0
0·505	—	0·045	10·4	24·2
0·758	—	—	6·3	34·0
1·01	—	—	2·7	24·2
1·52	—	—	—	7·8

$R_M = 5250$, $x/M = 40$

$\frac{k}{k_s}$	$\frac{\phi(k)}{\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\frac{k^2\phi(k)}{k_s^2\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\frac{k^4\phi(k)}{k_s^4\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ ($\times 10^3$)	$\frac{k^6\phi(k)}{k_s^6\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ ($\times 10^4$)
0·0070	65	—	—	—
0·0141	79	—	—	—
0·0211	75	—	—	—
0·0282	61	—	—	—
0·0352	55	—	—	—
0·0528	37	—	—	—
0·0704	27	0·123	0·73	—
0·106	14	0·150	1·95	—
0·176	6·5	0·156	4·72	1·9
0·246	—	0·124	7·4	4·6
0·352	—	0·084	9·4	10·0
0·528	—	—	7·8	19·5
0·704	—	—	5·1	20·4
1·06	—	—	1·5	12·3
1·41	—	—	—	3·2

$R_M = 5250$, $x/M = 60$

$\frac{k}{k_s}$	$\frac{\phi(k)}{\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\frac{k^2\phi(k)}{k_s^2\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\frac{k^4\phi(k)}{k_s^4\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ ($\times 10^3$)	$\frac{k^6\phi(k)}{k_s^6\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ ($\times 10^4$)
0·0091	50	—	—	—
0·0183	68	—	—	—
0·0274	61	—	—	—
0·0365	53	—	—	—
0·0456	46	—	—	—
0·0685	26	—	—	—
0·0914	18	0·139	1·18	—
0·137	10	0·165	2·83	—
0·228	4	0·153	7·1	3·6
0·320	—	0·089	9·4	10·3
0·456	—	0·052	9·6	10·2
0·685	—	—	5·7	28·3
0·914	—	—	2·8	24·9
1·37	—	—	—	6·2
1·83	—	—	—	3·1

$R_M = 5250$, $x/M = 80$

$\frac{k}{k_s}$	$\frac{\phi(k)}{\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\frac{k^2\phi(k)}{k_s^2\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\frac{k^4\phi(k)}{k_s^4\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ ($\times 10^3$)	$\frac{k^6\phi(k)}{k_s^6\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ ($\times 10^4$)
0·0108	47	—	—	—
0·0215	59	—	—	—
0·0323	54	—	—	—
0·0430	41	—	—	—
0·0538	35	—	—	—
0·0807	23	—	—	—
0·108	14	0·153	1·32	—
0·161	7	0·163	4·1	—
0·269	2·5	0·123	7·9	6·9
0·377	—	0·080	10·4	14·7
0·538	—	0·027	9·1	21·6
0·806	—	—	4·4	23·9
1·08	—	—	2·2	16·1
1·61	—	—	—	3·1

$R_M = 10500$, $x/M = 30$

$\dfrac{k}{k_s}$	$\dfrac{\phi(k)}{\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\dfrac{k^2\phi(k)}{k_s^2\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\dfrac{k^4\phi(k)}{k_s^4\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ $(\times 10^3)$	$\dfrac{k^6\phi(k)}{k_s^6\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ $(\times 10^4)$
0·0069	148	—	—	—
0·0137	150	—	—	—
0·0206	104	—	—	—
0·0275	84	—	—	—
0·0344	63	—	—	—
0·0515	41	—	—	—
0·069	30	0·129	0·82	—
0·103	14	0·161	1·51	—
0·172	6	0·151	3·90	1·54
0·240	—	0·112	6·4	3·7
0·344	—	0·063	7·3	7·4
0·515	—	—	5·7	13·9
0·687	—	—	3·5	14·5
1·03	—	—	1·0	8·2
1·37	—	—	—	1·9

$R_M = 10500$, $x/M = 40$

$\dfrac{k}{k_s}$	$\dfrac{\phi(k)}{\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\dfrac{k^2\phi(k)}{k_s^2\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\dfrac{k^4\phi(k)}{k_s^4\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ $(\times 10^3)$	$\dfrac{k^6\phi(k)}{k_s^6\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ $(\times 10^4)$
0·0084	143	—	—	—
0·0168	136	—	—	—
0·0252	91	—	—	—
0·0336	66	—	—	—
0·042	56	—	—	—
0·063	35	—	—	—
0·084	24	0·168	1·10	—
0·126	12	0·186	2·49	—
0·210	—	0·158	6·0	3·5
0·294	—	0·104	7·7	7·5
0·420	—	0·057	7·8	15·5
0·630	—	—	4·4	25·6
0·840	—	—	2·3	22·2
1·26	—	—	—	10·1
1·68	—	—	—	4·1

$R_M = 10500$, $x/M = 60$

$\dfrac{k}{k_s}$	$\dfrac{\phi(k)}{\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\dfrac{k^2\phi(k)}{k_s^2\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\dfrac{k^4\phi(k)}{k_s^4\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ $(\times 10^3)$	$\dfrac{k^6\phi(k)}{k_s^6\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ $(\times 10^4)$
0·0108	121	—	—	—
0·0216	108	—	—	—
0·0324	78	—	—	—
0·0432	59	—	—	—
0·0540	44	—	—	—
0·0810	25	—	—	—
0·108	15	0·186	4·0	—
0·162	7	0·198	8·5	—
0·270	—	0·135	10·4	8·1
0·378	—	0·075	11·0	13·6
0·540	—	0·039	7·3	24·8
0·810	—	—	5·3	28·8
1·08	—	—	2·5	16·9
1·62	—	—	—	5·6

$R_M = 21000$, $x/M = 30$

$\dfrac{k}{k_s}$	$\dfrac{\phi(k)}{\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\dfrac{k^2\phi(k)}{k_s^2\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$	$\dfrac{k^4\phi(k)}{k_s^4\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ $(\times 10^3)$	$\dfrac{k^6\phi(k)}{k_s^6\epsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}}$ $(\times 10^4)$
0·0081	199	—	—	—
0·0163	144	—	—	—
0·0244	92	—	—	—
0·0325	70	—	—	—
0·0406	59	—	—	—
0·0610	35	—	—	—
0·0812	21	0·148	1·55	—
0·122	11	0·163	2·88	—
0·203	—	0·124	4·78	2·5
0·284	—	0·079	5·65	5·0
0·406	—	0·036	5·71	9·1
0·610	—	—	3·30	12·2
0·813	—	—	1·48	10·4
1·22	—	—	—	3·4

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